An extended phase-space stochastic quantization of constrained Hamiltonian systems

2008 J. Phys. A: Math. Theor. 41 315303

(http://iopscience.iop.org/1751-8121/41/31/315303)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 315303 (8pp) doi:10.1088/1751-8113/41/31/315303

An extended phase-space stochastic quantization of constrained Hamiltonian systems

G T Ter-Kazarian[1] **and Y Sobouti**[2]

[1] Byurakan Astrophysical Observatory, Byurakan 378433, Aragatsotn District, Armenia
[2] Institute for Advanced Studies in Basic Sciences, Gava Zang, Zanjan, PO Box 45195-159, Iran

E-mail: gago-50@yahoo.com and sobouti@iasbs.ac.ir

Received 24 March 2008, in final form 11 May 2008
Published 4 July 2008
Online at stacks.iop.org/JPhysA/41/315303

Abstract
Having gained some insight into the concept of 'actual and virtual paths' in a phase-space formalism (Sobouti and Nasiri 1993 *Int. J. Mod. Phys.* B **7** 3255, Nasiri *et al* 2006 *J. Math. Phys.* **47** 092106), in the present paper we address the question of 'extended' phase-space stochastic quantization of Hamiltonian systems with first class holonomic constraints. We present the appropriate Langevin equations, which quantize such constrained systems, and prove the equivalence of the stochastic quantization method with the conventional path-integral gauge measure of Faddeev–Popov quantization.

PACS numbers: 02.50.Ey, 02.50.Ga, 03.65.Ta, 05.30.−d, 05.10.Gg

1. Introduction

The concept of an 'extended' Lagrangian, $\mathcal{L}(p, q, \dot{p}, \dot{q})$, in phase-space allows a subsequent extension of Hamilton's principle to minimum actions along the actual trajectories in (p, q) rather than in q-space. The following notational conventions are used throughout this paper: $\dot{p}$ denotes $\mathrm{d}p/\mathrm{d}x_0$ and so on, where x_0 is the real time. This extension, in turn, allows a definition of 'second' momenta $\pi_p = \delta\mathcal{L}/\delta\dot{p}$ and $\pi_q = \delta\mathcal{L}/\delta\dot{q}$, and a subsequent introduction of an 'extended' phase-space (p, q, π_p, π_q) and of an 'extended' Hamiltonian, $\mathcal{H}(p, q, \pi_p, \pi_q)$ [1]. This simple formalism manifests its practical and technical virtue in the proposed canonical quantization in (p, q) space that at once provides a framework for quantum statistical mechanics, for the classical statistical mechanics (Liouville's equation), for the conventional quantum mechanics as a special case, for von Neumann's density matrix and its equation of evolution as its inevitable corollaries. Wigner's [2] distributions and the equation satisfied by them are also obtained from those of [1] by an appropriate canonical transformation in the proposed (p, q, π_p, π_q)-space.

Ordering of p and q factors in conventional quantum mechanics has always been a matter of debate. For, there is nothing in the basic postulates of quantum mechanics to decide on

1751-8113/08/315303+08$30.00

the issue. On the other hand, the phase-space quantization is constructed on the premises that p and q are independent variables. Thus, in reducing the theory to that of Schrödinger and/or Heisenberg, the standard ordering emerges as the rule of game: for example, qp in q-representation and pq in p-representation. For Wigner's distributions the appropriate ordering is the symmetric one: for example, $\frac{1}{2}(pq + qp)$ instead of pq or qp. This ordering is also obtained from the standard ordering by the same canonical transformation which transforms the state functions and evolution equations of [1] to those of Wigner.

Since Wigner's initial attempt, 1932, alternative phase-space distributions have been proposed. Of these alternatives, the ones compatible with the uncertainty principle are obtainable from that of [1] by suitable canonical transformation in (p, q, π_p, π_q)-space. Husimi's all-positive distributions [3] are, however, exceptions. For example, his averaging of Wigner's distributions over small cells around phase-space points makes the averaged distributions incompatible with the uncertainty principle.

The stochastic quantization method (SQM) of recent years [4] is an alternative to the conventional canonical and path-integral quantizations. Conceptually and techniquewise it is versatile and powerful. Our interest here is to generalize SQM to study the classical stochastic processes underlying the phase-space quantization. In its present formulation, SQM exploits the well-defined Markoffian process of Wiener's type with Gaussian white noise. One may, however, envisage that different stochastic processes with respect to a fictitious time may yield different variations of quantum theories. In what follows, we give a SQM theory of the phase-space quantization. This paper consists of two parts. Section 2 deals with unconstrained systems and section 3 considers the constrained ones. An implicit summation on repeated indices is assumed.

2. An extended phase-space formulation of the SQM

Consider a dynamical system with N degrees of freedom described by the $2N$ coordinates $q = (q_1, \ldots, q_N)$, momenta $p = (p_1, \ldots, p_N)$, a Lagrangian $\mathcal{L}^q(q, \dot{q})$ in q-representation and the corresponding $\mathcal{L}^p(p, \dot{p})$ in p-representation. In general, $\mathcal{L}^q$ and $\mathcal{L}^p$ are the Fourier transforms of each other. In the framework of the proposed extended phase-space formalism of [1], the extended Lagrangian is written as

$$\mathcal{L}(p, q, \dot{p}, \dot{q}) = -\dot{q}_i p_i - q_i \dot{p}_i + \mathcal{L}^q + \mathcal{L}^p, \tag{1}$$

where p and q are independent and not, in general, canonical pairs. They could be so but only through a proper choice of the initial values. The first two terms in equation (1) constitute a total time derivative and are introduced for later convenience. The independent nature of p and q gives the freedom of introducing a second set of canonical momenta for both p and q through the extended Lagrangian of equation (1). Thus,

$$\pi_{q_i} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}^q}{\partial \dot{q}_i} - p_i, \qquad \pi_{p_i} = \frac{\partial \mathcal{L}}{\partial \dot{p}_i} = \frac{\partial \mathcal{L}^p}{\partial \dot{p}_i} - q_i. \tag{2}$$

Vanishing of π_q /or π_p is the condition for p and q to constitute a canonical pair. In the language of statistical quantum mechanics this choice picks up a pure state. Otherwise, one deals with a mixed state. One may now define an extended Hamiltonian

$$\begin{aligned} \mathcal{H}(p, q, \pi_p, \pi_q) &= \pi_{q_i}\dot{q}_i + \pi_{p_i}\dot{p}_i - \mathcal{L}(p, q, \dot{p}, \dot{q}) \\ &= H(p + \pi_q, q) - H(p, q + \pi_p) = \sum_{n=1} \frac{1}{n!}\left\{\frac{\partial^n H}{\partial p^n}\pi_q^n - \frac{\partial^n H}{\partial q^n}\pi_p^n\right\}, \end{aligned} \tag{3}$$

where $H(p,q) = p_i\dot{q}_i - \mathcal{L}^q = q_i\dot{p}_i - \mathcal{L}^p$ is the conventional Hamiltonian of the system. Introducing an imaginary time $x_4 = \mathrm{i}x_0$, we define the Euclidian extended action as

$$S[p,q,\pi_p,\pi_q] = \int \left[-\mathrm{i}\pi_{p_i}\frac{\mathrm{d}p_i}{\mathrm{d}x_4} - \mathrm{i}\pi_{q_i}\frac{\mathrm{d}q_i}{\mathrm{d}x_4} + \mathcal{H}(p,q,\pi_p,\pi_q)\right]\mathrm{d}x_4. \tag{4}$$

Following a general prescription of SQM [4], in the case at hand the Parisi–Wu ansatz consists of proposing a Markoffian hypothetical stochastic process by the following set of Langevin equations:

$$\begin{aligned} \frac{\mathrm{d}q_i}{\mathrm{d}t} &= -\frac{\delta S}{\delta q_i} + \xi_i^q(t), & \gamma\frac{\mathrm{d}\pi_{q_i}}{\mathrm{d}t} &= -\frac{1}{\gamma}\frac{\delta S}{\delta \pi_{q_i}} + \eta_i^q(t), \\ \gamma\frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\frac{1}{\gamma}\frac{\delta S}{\delta p_i} + \eta_i^p(t), & \frac{\mathrm{d}\pi_{p_i}}{\mathrm{d}t} &= -\frac{\delta S}{\delta \pi_{p_i}} + \xi_i^p(t), \end{aligned} \tag{5}$$

where an additional 'fictitious time' t is introduced, the $\xi_i^\nu(t)$ and $\eta_i^\nu(t)$ $(\nu = q, p)$ are Gaussian white-noise sources with

$$\begin{aligned} &\left\langle \xi_i^\nu(t), \xi_j^{\nu'}(t')\right\rangle = 2\delta_{ij}\delta_{\nu\nu'}\delta(t-t'), \qquad \left\langle \eta_i^\nu(t), \eta_j^{\nu'}(t')\right\rangle = 2\delta_{ij}\delta_{\nu\nu'}\delta(t-t'), \\ &\left\langle \xi_i^\nu(t), \eta_j^{\nu'}(t')\right\rangle = 0, \end{aligned} \tag{6}$$

and γ is an arbitrary dimensional parameter. In this case, we have only to look upon the fictitious time t as a mathematical tool, but need not find its physical meaning. A remark on notation: functional dependences on variables are indicated by square brackets, such as $S[p,q,\ldots]$. Functional derivatives are shown by δ' such as $\delta S/\delta p$, etc. The formalism being followed is based on a well-defined classical Wiener–Markoffian process. The Gaussian white noises incorporated into equation (5) are designed to yield the quantum mechanics as its thermal equilibrium limit. Therefore, the task is to show that the dynamical system described by equations (5) and (6) has an equilibrium distribution equivalent to the conventional path-integral measure. The procedure is (a) to define a Fokker–Planck Lagrangian based on equation (5), (b) to define Fokker–Planck momenta from this Lagrangian, (c) to compose a Fokker–Planck Hamiltonian and finally (d) to set up the Fokker–Planck equation for the distribution of the system in the extended phase-space. Thus

(a) The Fokker–Planck Lagrangian corresponding to equation (5) is

$$\begin{aligned} \mathcal{L}^F = \frac{1}{4}\sum_{i=1}^{N}\Bigg[&\left(\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\delta S}{\delta q_i}\right)^2 + \left(\gamma\frac{\mathrm{d}\pi_{q_i}}{\mathrm{d}t} + \frac{1}{\gamma}\frac{\delta S}{\delta \pi_{q_i}}\right)^2 \\ &+ \left(\gamma\frac{\mathrm{d}p_i}{\mathrm{d}t} + \frac{1}{\gamma}\frac{\delta S}{\delta p_i}\right)^2 + \left(\frac{\mathrm{d}\pi_{p_i}}{\mathrm{d}t} + \frac{\delta S}{\delta \pi_{p_i}}\right)^2\Bigg], \end{aligned} \tag{7}$$

where the first and fourth terms in the bracket originate from the white-noise sources $\xi_i^\nu(t)$ and the second and third originate from $\eta_i^\nu(t)$.

(b) The Fokker–Planck canonical momenta are

$$\begin{aligned}
\pi^F_{q_i} &= \frac{\partial \mathcal{L}^F}{\partial(\mathrm{d}q_i/\mathrm{d}t)} = \frac{1}{2}\left(\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\delta S}{\delta q_i}\right),\\
\pi^F_{\pi_{q_i}} &= \frac{\partial \mathcal{L}^F}{\partial\left(\mathrm{d}\pi_{q_i}/\mathrm{d}t\right)} = \frac{\gamma}{2}\left(\gamma\frac{\mathrm{d}\pi_{q_i}}{\mathrm{d}t} + \frac{1}{\gamma}\frac{\delta S}{\delta \pi_{q_i}}\right),\\
\pi^F_{p_i} &= \frac{\partial \mathcal{L}^F}{\partial(\mathrm{d}p_i/\mathrm{d}t)} = \frac{\gamma}{2}\left(\gamma\frac{\mathrm{d}p_i}{\mathrm{d}t} + \frac{1}{\gamma}\frac{\delta S}{\delta p_i}\right),\\
\pi^F_{\pi_{p_i}} &= \frac{\partial \mathcal{L}^F}{\partial\left(\mathrm{d}\pi_{p_i}/\mathrm{d}t\right)} = \frac{1}{2}\left(\frac{\mathrm{d}\pi_{p_i}}{\mathrm{d}t} + \frac{\delta S}{\delta \pi_{p_i}}\right).
\end{aligned} \tag{8}$$

(c) The Fokker–Planck Hamiltonian is

$$\mathcal{H}^F = \pi^F_{q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \pi^F_{\pi_{q_i}}\frac{\mathrm{d}\pi_{q_i}}{\mathrm{d}t} + \pi^F_{p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} + \pi^F_{\pi_{p_i}}\frac{\mathrm{d}\pi_{p_i}}{\mathrm{d}t} - \mathcal{L}^F. \tag{9}$$

(d) Finally, the Fokker–Planck equation for the probability distribution $\Phi[p, q, \pi_p, \pi_q, t]$ is

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi[p, q, \pi_p, \pi_q, t] &= \mathcal{H}^F\Phi[p, q, \pi_p, \pi_q, t]\\
&= \left[\frac{\partial}{\partial q_i}\left(\frac{\partial}{\partial q_i} + \frac{\delta S}{\delta q_i}\right) + \frac{1}{\gamma^2}\frac{\partial}{\partial \pi_{q_i}}\left(\frac{\partial}{\partial \pi_{q_i}} + \frac{\delta S}{\delta \pi_{q_i}}\right)\right.\\
&\quad \left. + \frac{1}{\gamma^2}\frac{\partial}{\partial p_i}\left(\frac{\partial}{\partial p_i} + \frac{\delta S}{\delta p_i}\right) + \frac{\partial}{\partial \pi_{p_i}}\left(\frac{\partial}{\partial \pi_{p_i}} + \frac{\delta S}{\delta \pi_{p_i}}\right)\right]\Phi[p, q, \pi_p, \pi_q, t].
\end{aligned} \tag{10}$$

Here we have replaced the 'canonical' momenta $\pi^F_{q_i}, \pi^F_{\pi_{q_i}}, \pi^F_{p_i}$ and $\pi^F_{\pi_{p_i}}$ with $-\partial/\partial q_i$, $-\partial/\partial \pi_{q_i}$, $-\partial/\partial p_i$ and $-\partial/\partial \pi_{p_i}$, respectively. The equilibrium distribution of equations (10) clearly reads

$$\Phi[p, q, \pi_p, \pi_q] \propto \exp(-S[p, q, \pi_p, \pi_q]). \tag{11}$$

Thus, the Langevin equation (5) together with equations (6) gives the same result as the conventional path-integral quantization method in the extended phase-space if only the drift forces

$$K_i(p, \ldots, t) = -\left(\frac{\delta S[p, \ldots]}{\delta p_i}\right)_{p=p(x_0,t)},$$

etc, have a damping effect. Along the actual trajectories in q-space equation (11) reproduces the results obtained in [4]. Along the trajectories in (p, q) space, however, it produces the state functions, $\chi(p, q, x_0)$, of [1]

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial x_0}\chi = \mathcal{H}\chi &= \left\{H\left(p - \mathrm{i}\hbar\frac{\partial}{\partial q}, q\right) - H\left(p, q - \mathrm{i}\hbar\frac{\partial}{\partial p}\right)\right\}\chi\\
&= \sum_{n=1}\frac{(-\mathrm{i}\hbar^n)}{n!}\left\{\frac{\partial^n H}{\partial p^n}\frac{\partial^n}{\partial q^n} - \frac{\partial^n H}{\partial q^n}\frac{\partial^n}{\partial p^n}\right\}\chi.
\end{aligned} \tag{12}$$

In order to obtain a feeling for this point we re-instate $\hbar$ for the rest of this section. Solutions of equation (12) are

$$\chi = a_{\alpha\beta}\psi_\alpha(q, x_0)\phi^*_\beta(p, x_0)\,\mathrm{e}^{-\mathrm{i}pq/\hbar}, \qquad a = a^\dagger, \qquad \text{positive definite}, \quad \mathrm{tr}\, a = 1, \tag{13}$$

where summation over repeated indices is implied, and ψ_α and ϕ^*_α are solutions of the conventional Schrödinger equation in q- and p-representations, respectively. They are mutually Fourier transforms

$$\begin{aligned}\psi_\alpha(q, x_0) &= (2\pi\hbar)^{-N/2} \int \phi_\alpha(p, x_0)\, \mathrm{e}^{\mathrm{i}pq/\hbar}\, \mathrm{d}p,\\ \phi_\alpha(p, x_0) &= (2\pi\hbar)^{-N/2} \int \psi_\alpha(q, x_0)\, \mathrm{e}^{-\mathrm{i}pq/\hbar}\, \mathrm{d}q.\end{aligned} \tag{14}$$

Note that the α and β are not, in general, eigenindices. The normalization condition for χ is

$$\int \chi\, \mathrm{d}p\, \mathrm{d}q = \mathrm{tr}(a) = 1. \tag{15}$$

See [1] for further details.

3. Stochastic quantization of extended dynamical systems with constraints

In this section we discuss the SQM of an extended dynamical system with M first class independent and irreducible constraints

$$\phi^a(p, q, \pi_p, \pi_q) = 0, \qquad a = 1, 2, \ldots, M < N. \tag{16}$$

For reasons of simplicity let there also be M gauge conditions:

$$\chi^a(p, q, \pi_p, \pi_q) = 0, \qquad a = 1, 2, \ldots, M. \tag{17}$$

Equations (16) and (17) define a $(4N - 2M)$ dimensional submanifold in phase-space on which the system orbits dwell. For convenience we introduce the following new variables:

$$x_{q_i} = (q_1, \ldots, q_N, \pi_{p_1}, \ldots, \pi_{p_N}), \qquad x_{p_i} = (p_1, \ldots, p_N, \pi_{q_1}, \ldots, \pi_{q_N}). \tag{18}$$

The gauge conditions are such that $\det \Delta^{ab} \neq 0$, where Δ^{ab} is the Poisson bracket of χ^a and ϕ^b,

$$\Delta^{ab} = \frac{\partial \chi^a}{\partial x_{q_i}} \frac{\partial \phi^b}{\partial x_{p_i}} - \frac{\partial \chi^a}{\partial x_{p_i}} \frac{\partial \phi^b}{\partial x_{q_i}}. \tag{19}$$

The Euclidean path-integral measure for such a system can be obtained by the quantization procedure of [5]. The Faddeev–Popov path-integral formula for this system is

$$\langle f|\mathrm{i}\rangle = \frac{1}{2N} \int \mathcal{D}x_{q_i} \mathcal{D}x_{p_i} \delta(\phi^a) \delta(\chi^a) \det \Delta^{ab} \exp(-S[x_p, x_q]), \tag{20}$$

where $S[x_p, x_q]$ is the extended Euclidian action of equation (4). Our major goal is now to reproduce equation (20) from the standpoint of SQM in phase-space in the thermal equilibrium limit. The Langevin equations for this system are

$$\begin{aligned}\frac{\mathrm{d}x_{q_i}}{\mathrm{d}t} &= -\frac{\delta S}{\delta x_{q_i}} - \left(\lambda^a \frac{\partial \phi^a}{\partial x_{p_i}} - \nu^a \frac{\partial \chi^a}{\partial x_{p_i}}\right) + \xi_i, \qquad \xi_i = (\xi_i^q, \xi_i^p),\\ \frac{\mathrm{d}x_{p_i}}{\mathrm{d}t} &= -\frac{\delta S}{\delta x_{p_i}} + \left(\lambda^a \frac{\partial \phi^a}{\partial x_{q_i}} - \nu^a \frac{\partial \chi^a}{\partial x_{q_i}}\right) + \eta_i, \qquad \eta_i = (\eta_i^p, \eta_i^q),\end{aligned} \tag{21}$$

where $\lambda^a(x_q, x_p)$ and $\nu^a(x_q, x_p)$, $a = 1, \ldots, M$, are $2M$ Lagrange multiplier functions. They are introduced to make provisions for the forces arising from the constraints and the gauge conditions. To eliminate the Lagrange multipliers we transform from $(x_{qi}, x_{pi}; i = 1, \ldots, 2N)$ to the new variables $(Q^i, P^i; i = 1, \ldots, 2N)$ such that

$$Q^a = \phi^a, \qquad P^a = \chi^a, \qquad a = 1, \ldots, M \tag{22}$$

and determine the remaining ones from the following differential equations:

$$\begin{aligned}
&\frac{\partial Q^\alpha}{\partial x_{q_i}}\frac{\partial \phi^a}{\partial x_{q_i}} + \frac{\partial Q^\alpha}{\partial x_{p_i}}\frac{\partial \phi^a}{\partial x_{p_i}} = 0, \qquad \frac{\partial Q^\alpha}{\partial x_{q_i}}\frac{\partial \chi^a}{\partial x_{q_i}} + \frac{\partial Q^\alpha}{\partial x_{p_i}}\frac{\partial \chi^a}{\partial x_{p_i}} = 0,\\
&\frac{\partial P^\alpha}{\partial x_{q_i}}\frac{\partial \phi^a}{\partial x_{q_i}} + \frac{\partial P^\alpha}{\partial x_{p_i}}\frac{\partial \phi^a}{\partial x_{p_i}} = 0, \qquad \frac{\partial P^\alpha}{\partial x_{q_i}}\frac{\partial \chi^a}{\partial x_{q_i}} + \frac{\partial P^\alpha}{\partial x_{p_i}}\frac{\partial \chi^a}{\partial x_{p_i}} = 0.
\end{aligned} \tag{23}$$

0.8cm $a = 1, \ldots, M$; $\alpha = M+1, \ldots, 2N$. Hereafter, we will reserve the subscripts a, b, etc, for indices ranging from 1 to M; α, β, etc, for those ranging from $M+1$ to $2N$; and i, j, etc for the whole range, 1 to $2N$. The first $2M$ variables of equation (22) are, by equations (16) and (17), constants along the phase-space trajectories of the system and satisfy

$$\frac{\mathrm{d}Q^a}{\mathrm{d}t} = \frac{\partial Q^a}{\partial x_{q_i}}\frac{\mathrm{d}x_{q_i}}{\mathrm{d}t} + \frac{\partial Q^a}{\partial x_{p_i}}\frac{\mathrm{d}x_{p_i}}{\mathrm{d}t} = 0, \tag{24}$$

$$\frac{\mathrm{d}P^a}{\mathrm{d}t} = \frac{\partial P^a}{\partial x_{q_i}}\frac{\mathrm{d}x_{q_i}}{\mathrm{d}t} + \frac{\partial P^a}{\partial x_{p_i}}\frac{\mathrm{d}x_{p_i}}{\mathrm{d}t} = 0. \tag{25}$$

To obtain the Langevin equations for $\mathrm{d}Q^\alpha/\mathrm{d}t$ and $\mathrm{d}P^\alpha/\mathrm{d}t$ we express them in terms $(\mathrm{d}x_{q_i}/\mathrm{d}t)$; and $(\mathrm{d}x_{p_i}/\mathrm{d}t)$ as in equations (24) and (25), and substitute for the latter from equations (21). The Lagrange multipliers drop out on account of equations (24) and (25). We arrive at

$$\begin{aligned}
\frac{\mathrm{d}Q^\alpha}{\mathrm{d}t} = &-\left(\frac{\partial Q^\alpha}{\partial x_{q_i}}\frac{\partial Q^\beta}{\partial x_{q_i}} + \frac{\partial Q^\alpha}{\partial x_{p_i}}\frac{\partial Q^\beta}{\partial x_{p_i}}\right)\frac{\delta S}{\delta Q^\beta}\\
&-\left(\frac{\partial Q^\alpha}{\partial x_{q_i}}\frac{\partial P^\beta}{\partial x_{q_i}} + \frac{\partial Q^\alpha}{\partial x_{p_i}}\frac{\partial P^\beta}{\partial x_{p_i}}\right)\frac{\delta S}{\delta P^\beta} + \frac{\partial Q^\alpha}{\partial x_{q_i}}\xi_i + \frac{\partial Q^\alpha}{\partial x_{p_i}}\eta_i,
\end{aligned} \tag{26}$$

$$\begin{aligned}
\frac{\mathrm{d}P^\alpha}{\mathrm{d}t} = &-\left(\frac{\partial P^\alpha}{\partial x_{q_i}}\frac{\partial Q^\beta}{\partial x_{q_i}} + \frac{\partial P^\alpha}{\partial x_{p_i}}\frac{\partial Q^\beta}{\partial x_{p_i}}\right)\frac{\delta S}{\delta Q^\beta}\\
&-\left(\frac{\partial P^\alpha}{\partial x_{q_i}}\frac{\partial P^\beta}{\partial x_{q_i}} + \frac{\partial P^\alpha}{\partial x_{p_i}}\frac{\partial P^\beta}{\partial x_{p_i}}\right)\frac{\delta S}{\delta P^\beta} + \frac{\partial P^\alpha}{\partial x_{q_i}}\xi_i + \frac{\partial P^\alpha}{\partial x_{p_i}}\eta_i.
\end{aligned} \tag{27}$$

Due to conditions of equations (23), the terms with $i = a = 1, \ldots, M$, do not contribute to equations (26) and (27). Therefore, the sum over i is replaced by the sum over β. Next we attempt to write equations (23)–(27) in a covariant form; that is, in a form invariant under general coordinate transformations. We introduce the notation

$$(x^I) = (x_I) = (x_{qi}, x_{pi}), \qquad (X^I) = (Q^a, P^a, Q^\alpha, P^\alpha), \qquad I = 1, \ldots, 4N. \tag{28}$$

Hereafter, the following convention will be observed in indexing the new variables. To begin with, the x-coordinates are Euclidean ones. It will not matter if they are indexed covariantly or contravariantly. The X-coordinates, on the other hand, are curvilinear ones. A contravariant index could be lowered by an appropriate metric tensor to be introduced shortly. As indicated by the defining equation (28), the manifold M^{4N} spanned by X-coordinates could be split into two submanifolds M^{2M} and M^{4N-2M}. The X-coordinates spanning M^{2M} will be indexed by $A, B, \ldots = 1, \ldots, 2M$. Those spanning M^{4N-2M} will be indexed by $\Lambda, \Sigma, \ldots = 2M+1, \ldots, 4N$. The indices $I, J, \ldots$, will be reserved for the whole manifold M^{4N}. The contravariant metric tensor for the curvilinear X-coordinates is, by equation (28),

$$g^{IJ} = \frac{\partial X^I}{\partial x^K}\frac{\partial X^J}{\partial x_K} = \begin{bmatrix} g^{AB} & \\ & g^{\Lambda\Sigma} \end{bmatrix}, \tag{29}$$

where the $2M \times 2M$ tensor g^{AB} is

$$g^{AB} = \frac{\partial X^A}{\partial x^K}\frac{\partial X^B}{\partial x_K} = \begin{bmatrix} \{\phi^a, \phi^b\} & \{\phi^a, \chi^b\} \\ \{\chi^a, \phi^b\} & \{\chi^a, \chi^b\} \end{bmatrix}. \tag{30}$$

The Poisson brackets, here, are to be calculated in (x_{q_i}, x_{p_i})-coordinates. For conventional gauge conditions one has $\{\chi^a, \chi^b\} = 0$. The Laplace expansion of det g^{AB} then gives

$$\det g^{AB} = -\det\{\phi^c, \chi^d\}\det\{\chi^e, \phi^f\} = \det\{\phi^c, \chi^d\}^2, \qquad \det g_{AB} = \det\{\phi^c, \chi^d\}^{-2}. \tag{31}$$

The expression for $g^{\Lambda\Sigma}$ is

$$g^{\Lambda\Sigma} = \frac{\partial X^\Lambda}{\partial x^K}\frac{\partial X^\Sigma}{\partial x_K}, \qquad \Lambda, \Sigma = 2M+1, \ldots, 4N. \tag{32}$$

At present, there is no need to manipulate $g^{\Lambda\Sigma}$ beyond its definition. We are now in a position to write the Langevin equations in manifest covariant forms. Equations (23) become

$$\frac{\mathrm{d}X^\Lambda}{\mathrm{d}x^I}\frac{\mathrm{d}X^A}{\mathrm{d}x_I} = 0. \tag{33}$$

Equations (24) and (25) become

$$\frac{\mathrm{d}X^A}{\mathrm{d}t} = 0. \tag{34}$$

Equations (26) and (27), combined together, give

$$\frac{\mathrm{d}X^\Lambda}{\mathrm{d}t} = -g^{\Lambda\Sigma}\frac{\delta\widetilde{S}[X]}{\delta X^\Sigma} + \frac{\partial X^\Lambda}{\mathrm{d}x^I}\zeta^I, \tag{35}$$

where $\zeta^I = (\xi^i, \eta^j)$, and $\widetilde{S}[X] = S[x(X)]$ is the action integral of equation (4) written in X-coordinates. Equation (34) contains no new information, beyond the fact that ϕ^a, χ^a are to vanish along the phase-space trajectories. Equation (35) holds on the constraint surface, M^{4N-2M}. Finally the form-invariant Fokker–Planck equation corresponding to equation (35) emerges as

$$\frac{\partial\widetilde{\Phi}[X^\Lambda, t]}{\partial t} = \frac{1}{\sqrt{\det g^{\Lambda\Sigma}}}\frac{\partial}{\partial X^\Lambda}\left[\sqrt{\det g^{\Lambda\Sigma}}\, g^{\Lambda\Sigma}\left(\frac{\partial}{\partial X^\Sigma} + \frac{\delta\widetilde{S}[X]}{\delta X^\Sigma}\right)\widetilde{\Phi}[X^\Lambda, t]\right]. \tag{36}$$

The stationary solution of equation (36) is

$$\widetilde{\Phi}_{eq}[X^\Lambda] = \frac{1}{\sqrt{\det g^{\Lambda\Sigma}}}\exp(-\widetilde{S}[X^\Lambda]). \tag{37}$$

In the limit of thermodynamic equilibrium the probability of finding the system on the constraint surface in the volume element $d^{(4N-2M)}X$ centred at X^Λ is

$$\frac{1}{\sqrt{\det g^{\Lambda\Sigma}}}\exp(-\widetilde{S}[X^\Lambda])d^{(4N-2M)}X = \frac{1}{\sqrt{\det g^{\Lambda\Sigma}}}\exp(-\widetilde{S}[X^I])\delta^{(2M)}(X^A)d^{(4N)}X, \tag{38}$$

where $\delta^{(2M)}(X^A)$ is the Dirac delta function in M^{2M}. It is introduced to ensure that the system stays on the constraint surface.

To transform equation (38) back to the Euclidean coordinates $\{x^I\}$ we note that $\widetilde{S}[X]$ transform into $S[x]$, and the volume element $d^{(4N)}X$ transforms into $\sqrt{\det g^{IJ}}d^{(4N)}x$, where, from equation (29),

$$\det(g^{IJ}) = \det(g^{AB})\det(g^{\Lambda\Sigma}). \tag{39}$$

Substitution of equations (39) and (31) into equation (38) gives Φ_q in x-coordinates. Hence

$$\Phi_{eq}[x_p, x_q] = \det\{\chi^a, \Phi^b\}\delta^{(M)}(\chi^c)\delta^{(M)}(\Phi^d)\exp(-S[x_p, x_q]). \tag{40}$$

4. Conclusions

It is worth reflecting briefly upon the results obtained so far. The new conceptual element in the extended phase-space formulation is noteworthy. Extended canonical transformations allow us to go from one extended phase-space to another. This unifying feature of the theory makes the comparison of the various functions existing in the literature possible and transparent. We have developed the SQM in extended phase-space and shown how this method can be generalized to deal with systems subjected to first class constraints. We have proved that Lagrange's method of undermined multipliers yields the quantization of constrained systems in SQM and, in a natural way, results in the Faddeev–Popov conventional path-integral measure for gauge systems. One of the most remarkable features of the SQM is that one may quantize even dynamical systems with non-holonomic constraints, as is seen in the case of the stochastic gauge fixing.

Acknowledgments

The helpful and knowledgable comments from the anonymous referees are much appreciated. GT acknowledges the FAR/ANSEF's support of this research.

References

[1] Sobouti Y and Nasiri S 1993 *Int. J. Mod. Phys.* B **7** 3255
Nasiri S *et al* 2006 *J. Math. Phys.* **47** 092106
[2] Wigner E P 1932 *Phys. Rev.* **40** 749
[3] Husimi K 1940 *Proc. Phys. Math. Soc. Jpn.* **22** 264
[4] Chaturvedi S *et al* 1985 *Phys. Lett.* B **157** 400
Ryang S *et al* 1985 *Prog. Theor. Phys. Lett.* **73** 1295
Horowitz A M 1985 *Phys. Lett.* B **156** 89
Ohba I 1987 *Prog. Theor. Phys.* **77** 1267
[5] Parisi G and Wu Y-S 1981 *Sci. Sin.* **24** 483
Klauder J R 1983 *Acta Phys. Austrica Suppl.* **25** 251
Okano K 1984 *Memories the School Sci. Eng. Waseda Univ.* **48** 23
Seiler E 1984 *Acta Phys. Austrica Suppl.* **26** 259
Sakita B 1985 *Quantum Theory of Many Variable Systems and Fields* (Singapore: World Scientific)
Migdal A A 1986 *Usp. Fiz. Nauk* **149** 3
Migdal A A 1986 *Sov. Phys.—Usp.* **29** 389
Damgaard P H and Hüffel H 1987 *Phys. Rep.* **152** 227
Damgaard P H and Hüffel H 1988 *Stochastic Quantization* (selected papers) (Singapore: World Scientific)
Chaturvedi S *et al* 1990 *Stochastic Quantization of Parisi and Wu* (Napoli: Bibliopolis)
Namiki M 1992 *Stochastic Quantization* (Heidelberg: Springer)
[6] von Neumann J 1932 *Mathematische Grundlagen der Quanten Mechanik* (Berlin: Springer)
[7] Nelson E 1966 *Phys. Rev.* **150** 107
[8] Suzuki M 1976 *Prog. Theor. Phys.* **56** 1454
Suzuki M 1976 *Commun. Math. Phys.* **51** 183
Polonyi J and Wyld H W 1983 *Phys. Rev. Lett.* **51** 2257
[9] Namiki M *et al* 1984 *Prog. Theor. Phys.* **72** 350
[10] Faddeev L D and Popov V N 1967 *Phys. Lett.* **25B** 29
Faddeev L D 1969 *Teor. Mat. Fiz.* **1** 3